Epsilon-Stability in School Choice

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Abstract

In many school choice practices, scores, instead of ordinal rankings, are used to indicate students' qualification. We study school choice problems where students have ordinal preference over schools while their priorities at schools are in the form of cardinal scores. The cardinality of scores allows us to measure the intensity of priority violations and hence relax stability by proposing epsilon-stability. We also propose the epsilon-EADA mechanism to select the constrained efficient matching under epsilon-stability.

Keywords: School choice; Deferred acceptance algorithm; Stability; Pareto efficiency

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1 Introduction

School choice programs are designed to assign school seats to students. In recent school choice reforms, the student-proposing deferred acceptance algorithm (henceforth, DA;

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Gale and Shapley, 1962) has been gradually replacing existing mechanisms at many cities and school districts.[1] DA has the good properties of being optimally stable (Gale and Shapley, 1962) and strategy-proof (Dubins and Freedman, 1981; Roth, 1982), so that no student-school pair would rather being matched with each other and no student has incentive to misreport her preference.

However, the DA assignment is not always Pareto efficient for the students (Ergin, 2002). This is mainly due to the incompatibility of stability and efficiency. Another source of inefficiency is due to the exogenous tie-breaking rules used to break ties when different students may have the same priority at a school. Such tie-breaking is necessary for the running of DA. Using data from New York high school admission, Abdulkadiroğlu, Pathak and Roth (2009) show that the inefficiency from both sources are significant in practice.

We study the Pareto improvement of students' assignment when schools have cardinal priorities. In practice, the priority structures at schools are often exogenously determined by law or policy. In many districts such priorities are generated from weighted scores assigned to students. For instance, in France, the priorities at secondary schools are based on scores that depend on a combination of geographic, academic and social factors. In China, priorities for schools are usually determined by the scores of entrance examination. If we interpret scores as cardinal priorities, then they reflect the intensity of priorities. As a result, when priority violation happens, we can measure the intensity of the violation. Suppose i, j, and k are three students, all prefer attending some school. Their scores for this school are $90, 72$, and 70, respectively. If k is assigned but i and j are not, then k violates j's priority by an intensity of 2, and violates that of i's by 20. The violation on i's priority is much severe.

In school choice with weak priorities, if two students have the same priority at a school, one is assigned to this school, the other desires but is not assigned, it is not viewed

[1] Abdulkadiroğlu and Sönmez (2003) first suggested to use alternative mechanisms to replace the Boston mechanism. See Pathak (2016) for detailed survey on school choice reforms.

as a priority violation (Erdil and Ergin, 2008; Abdulkadiroğlu, Pathak and Roth, 2009). We extend this intuition to school choice with cardinal priorities and consider minor violations of priorities. We say that a matching is ε-stable if whenever student j violates student i's priority at a school, the score of j at this school is no less than that of i's minus ε. When ε is small, the matching is then almost stable. We illustrate with example that even with a small ε, ε-stable matchings may be significantly more efficient (in terms of students' welfare) than stable matchings.

We then propose an algorithm, the ε-EADAM, which is a variation of the EADAMs due to Kesten (2010) and Tang and Yu (2014), to produce ε-stable matching that is constrained efficient for school choice with cardinal priorities. An ε-stable matching is said to be constrained efficient if it is not Pareto dominated by any ε-stable matching. This algorithm starts by running DA, then iteratively reruns DA after removing students matched with underdemanded schools together with their assignments. In particular, for each student removed and any school that she desires, the algorithm forbids students with lower and not within ε-difference scores from applying to that school. Therefore, after the intensity of permitted violations is set as ε, the ε-EADAM procedure endogenously selects which close priorities need to be violated (or reshuffled) and obtains the most favorable outcome.

Our study is mainly motivated by the trade-off between stability and efficiency, in school choice practices where a cardinal and observable measure of priority violation is readily available. Such trade-off has been extensively discussed and evaluated in the literature (see, e.g., Abdulkadiroğlu, Pathak and Roth, 2009 and Pathak, 2016). Therefore, ε-stability can be useful if policy makers care not only about stability but also its trade-off with students' welfare, or if they care about absolute stability but there is common understanding that there are measurement errors in the scores. The latter could happen, for example, when scores are partly determined by exam grades, which only imperfectly reflect students' true ability. In that case, students will be less justified to claim that they are more qualified than those with slightly lower grades. Such observation has been made by Wu and Zhong (2014) and Lien, Zheng and Zhong (2016), who study a notion of ex-ante fairness based on students' true abilities instead of score-induced priorities, and use it as a criterion to compare different mechanisms. Accordingly, we leave the choice of ε for debate among practitioners. Our simple suggestion is that, ε should increase as the gains of students' welfare or the variance of the measurement errors increases.

Our relaxation of stability takes a similar form as that of Che, Kim and Kojima (2015), who study stability in large economies in a firm-worker set-up and also use ε-stability as an approximation of stability. Due to the similarity of minor priority difference and priority indifference, our study also closely relates to the study of school choice with weak priorities. Abdulkadiroğlu, Pathak and Roth (2009) offer both theoretical and empirical analysis on the consequence of weak priorities, and show that the efficiency loss in DA can be significant in practice. To recover the efficiency loss due to tie-breaking on weak priorities, Erdil and Ergin (2008) propose the stable improvement cycles algorithm (SIC), and Kesten (2010) proposes a variation of EADAM which is later simplified by Tang and Yu (2014). The SIC approach and EADAM approach differ in that the former leaves open on how to select improvement cycles when facing multiplicity (e.g., randomly pick one would be a default choice), while the latter selects cycles in a specific way by iteratively rerunning DA, and along the way, correcting the initial tie-breaker.[2]

Epsilon-stability most closely relates to the notion of sticky-stability proposed by Afacan et al. (2016). The two concepts share the same spirit but differ in the ways that stability is relaxed. Sticky-stability is defined by assuming that it is costly for students to appeal for schools at which their priorities are violated, hence students may decide not to appeal for schools that are not significantly better than their current assignments. Interpreted in terms of consenting constraint, in short, sticky-stability assumes that due to appealing cost, student i consents to give up her priority at a school s if s is only slightly better than her current assignment; by doing that, all students with lower s-priority than

[2]For more details on the connection between SIC and the simplified EADAM, see Tang and Yu (2014), section 4.2.

i are allowed to violate i's priority at s. Epsilon-stability, instead, assumes that student i consents to give up her priorties at all schools better than her current assignment, but for each such school s, only students with slightly lower priorities (within ε-difference in scores) are allowed to violate i's priority at s. In the language of Afacan et al., if we understand priority lists as schools' preferences, then ε-stability corresponds to the stickiness of schools' (instead of students') appealing incentives: if a student with only slightly lower score than i violates i's priority at school s, s may not want to appeal to be matched with i.

A subtle point worth mentioning is that the checking of ε-stability is more objective than that of sticky-stability. In practice, when schools' cardinal preferences are represented by scores, such scores are often observable. So if i's priority is violated at s, the difference between the score of the matched student and that of i (which will be compared with ε) is also observable. On the other hand, students' cardinal (or ordinal) preferences are subjective and need to be reported. When i's priority is violated at s, the difference cardinal utility (or rank) between s and her assignment is also subjective, so will the checking of sticky-stability. If i is not willing to consent, she will have incentive to exaggerate the difference in her report.

Afacan et al. (2016) also provide an algorithm which combines Kesten's EADAM and Erdil and Ergin's SIC algorithm to find a constrained efficient sticky-stable matching. Their algorithm can also be simplified by using Tang and Yu's simplified EADAM as we do here. To do that, after the removal of each student matched with underdemanded school, if any school is better than her assignment by certain rank, then all students with lower priorities than her at that school are forbidden from applying when rerunning DA.[3]

This paper is organized as follows. We introduce basic notations and concepts of the model in Section 2. In Section 3, we propose ε-stability and the ε-EADAM, and state our main results. We conclude in Section 4 and relegate all proofs into the Appendix.

[3]See Afacan et al. (2016), Proposition 3, for an example which illustrates of why a SIC stage is needed after applying Kesten's EADAM. The same example can also be used to illustrate why constrained efficiency can be achieved by using the simplified EADAM alone.

2 Preliminaries

A finite set of students $I \equiv \{1, 2, \cdots, n\}$ are to be matched with a finite set of schools $S \equiv \{s_1, s_2, \cdots, s_m\}$, where each school can admit students up to its capacity. Let q_s denote the capacity of school s. If a student is not matched with any school, we say that she is matched with the null school $\emptyset$, which has unlimited capacity. Every student $i \in I$ has a strict preference P_i over the set $S \cup \{\emptyset\}$. If student i prefers school s' to school s, we write it as $s' P_i s$. Let R_i denote the weak extension of P_i.

In practice, it is common that schools evaluate students by scores. Let $a_i^s \in [0, 100]$ denote student i's score at school s, where higher score means higher (cardinal) priority. Let $a^s \equiv (a_1^s, a_2^s, ..., a_n^s)$ denote the score vector for school s. If for some ordinal (weak) priority list $\succeq_s$ on students, i is weakly preferred to j (written as $i \succeq_s j$) if and only if $a_i^s \geq a_j^s$, then we say $\succeq_s$ is induced by the score vector a^s, or a^s represents $\succeq_s$.

A *matching* is a mapping $\mu : I \to S \cup \{\emptyset\}$ such that $|\mu^{-1}(s)| \leq q_s, \forall s \in S$. Matching μ *weakly Pareto dominates* matching ν if for all $i \in I$, $\mu(i) R_i \nu(i)$. Matching μ *Pareto dominates* ν if μ weakly Pareto dominates ν and $\mu \neq \nu$. We say a matching is *Pareto efficient* (or simply *efficient*) if it is not Pareto dominated by any other matching. A student i *desires* school s at matching μ if $s P_i \mu(i)$.

The priority of student i at school s is *violated* by student j if i desires s, $\mu(j) = s$, and $a_i^s > a_j^s$. A matching is *fair* if no student's priority at any school is violated. A matching is *non-wasteful* if whenever a school $s \in S \cup \{\emptyset\}$ is desired, $|\mu^{-1}(s)| = q_s$. We say a matching μ is *stable* if it is fair and non-wasteful.

A *school choice problem (with cardinal priorities)* is a pair (P, A), where $P \equiv (P_i)_{i \in I}$ and $A \equiv (a^s)_{s \in S}$, consists of a preference profile and a score matrix. A *mechanism* φ selects a matching $\varphi(P, A)$ for each school choice problem (P, A). A mechanism is stable (Pareto efficient, resp.) if it always selects stable (Pareto efficient, resp.) matchings.

3 Epsilon stability

3.1 ε-stable matching

Given a matching μ, a student i is said to desire a school s at μ if $sP_i\mu(i)$.

Definition 1. *A matching μ is ε-stable if it is*

(i) *ε-fair: there do not exist students i, j, and school s such that i desires s, $\mu(j) = s$, but $a_j^s < a_i^s - \varepsilon$; and*

(ii) *nonwasteful: s is desired implies $|\mu^{-1}(s)| = q_s$.*

We see that when $\varepsilon = 0$, ε-stability reduces to the usual notion of stability under possibly weak priorities. Let $M_\varepsilon(P, A)$ denote the set of ε-stable matchings for the problem (P, A). Fixing (P, A) and ε, we say a matching $\mu \in M_\varepsilon(P, A)$ is *ε-constrained efficient* if it is not Pareto dominated by any matching in $M_\varepsilon(P, A)$.

Example 1. The set of students and schools are $I \equiv \{i, j, k\}$ and $S \equiv \{s_1, s_2\}$, respectively, and each school has exactly one seat. The score matrix and the preference profile are given as follows:

	a^{s_1}	a^{s_2}
i	80	70
j	72	$\vdots$
k	70	80

P_i	P_j	P_k
$\boxed{s_2}$	s_1	$\boxed{s_1}$
$\underline{s_1}$	$\boxed{\underline{\emptyset}}$	$\underline{s_2}$

Let μ^1 and μ^2 denote the underlined and boxed matching in the preference table, respectively. μ^1 is produced by the student-proposing DA algorithm and hence is optimally stable, and is therefore ε-constrained efficient for $\varepsilon = 0$.

μ^2 is obtained from μ^1 by letting student i and k trade seats, and μ^2 violates j's priority at s_1 by an intensity of $72 - 70 = 2$. If $\varepsilon \geq 2$, then μ^2 is ε-stable and is ε-constrained efficient.

3.2 ε-EADAM

Fixing ε, a mechanism φ is constrained efficient if it selects a constrained efficient matching for every problem. In below we propose the ε-EADA mechanism to select ε-constrained efficient matchings for school choice problems with cardinal priorities.

The benchmark mechanism, which is widely used in school choice programs, is known as the Gale-Shapley *student-proposing deferred acceptance* (DA) algorithm due to Gale and Shapley (1962).

For each school choice problem $(P, \succ)$, DA operates as follows:

Step 1 Each student applies to her most desirable school. Each school tentatively accepts the best students according to its priority list, up to its capacity, and rejects the rest.

Step $k, k \geq 2$ Each student rejected in the previous round applies to her next best school. Each school that faces new applicants tentatively accepts the best students according to its priority list, up to its capacity, and rejects the rest, among both new applicants and previously accepted students.

The algorithm stops when no more student is rejected.

The efficiency-adjusted deferred acceptance mechanism (henceforth, EADAM) is proposed by Kesten (2010) to recover the inefficiency of DA, and is later simplified by Tang and Yu (2014).[4] Our ε-EADAM is an adaptation of the simplified EADAM.

The key concept in the definitions of the simplified EADAM and our adaptation is underdemanded school. Following Kesten and Kurino (2013) and Tang and Yu (2014), a school is said to be *underdemanded* at matching μ if no student desires this school at μ. It is not difficult to see that students matched to underdemanded schools are not Pareto improvable. Also, a school is underdemanded at the DA matching if and only if it never rejected any student during the DA procedure.

[4] Bando (2014) independently proposes another simplification of Kesten (2010), which is embedded in Tang and Yu (2014).

We now adapt the simplified EADAM to select ε-constrained efficient matchings for every school choice problems with cardinal priorities.

For every school choice problem with cardinal priorities (P, A) and every $\varepsilon > 0$, the *ε-EADAM* operates as follows:

Round 0. Randomly draw a tie-breaker to break ties and translate the score matrix A into a strict priority profile $\succ$. Run DA for the problem and obtain the matching $\mu^0 = DA(P, \succ)$.

Round $k, k \geq 1$**.** This round consists of three steps:

1. At matching μ^{k-1}, settle the matching at all underdemanded schools as it is and remove them together with the students matched with them.
2. For each student i removed, each remaining school s that i desires, and each remaining student j such that $a_j^s < a_i^s - \varepsilon$, remove s from j's preference.
3. For the remaining schools and students with the new preference profile, rerun DA (the round-k DA) to obtain the matching μ^k.

   The algorithm terminates when all schools have been removed.

Essentially, ε-EADAM can be viewed as an adaptation of the simplified EADAM which allows for more general consenting constraint. When a student becomes Pareto unimprovable, instead of consenting to give up her priorities at schools that she desires to all students with lower priorities than her, in ε-EADAM, she consents only to students whose scores are lower but are within ε difference from hers.

The following lemma could be proved following the same steps as in Tang and Yu (2014). We omit the proof here.

Lemma 1. *For each $k \geq 1$, the round-k DA matching of the ε-EADAM weakly Pareto dominates that of round-$(k-1)$. The ε-EADAM is well-defined and stops within $|S \cup \{\emptyset\}| + 1 = m + 2$ rounds.*

Therefore, as a mechanism, ε-EADAM Pareto dominates the student-proposing DA. Since no strategy-proof mechanism Pareto improves on DA(see, e.g., Kesten, 2010, or Abdulkadiroğlu, Pathak and Roth, 2009) , we know that ε-EADAM is not strategy-proof. Nonetheless, due to simiar arguments as Kesten (2010), this does not imply that ε-EADAM is easily manipulable in practice.[5]

In below we present our main result, which states that the ε-EADAM outcome is constrained efficient within the set of ε-stable matchings.

Theorem 1. *The ε-EADAM is ε-constrained efficient.*

The theorem can be viewed as a generalization of the main theorem of Tang and Yu (2014). We relegate its proof into the Appendix. The intuition behind the proof is that in each round of ε-EADAM, students who become Pareto unimprovable in the previous round DA are removed, hence their applications to schools they still desire are withdrawed. By doing so, the assignments of other students are weakly improved as some blockings of their exchanges are now relaxed. Furthermore, in each round, students' preferences are modified to ensure no priority violation exceeds ε.

Next, we present an example to illustrate how ε-EADAM works. Also, from this example, we see that as ε increases, it is possible that not all students become weakly better off.

Example 2. Consider $I \equiv \{1, 2, 3, 4\}$ and $S \equiv \{s_1, s_2, s_3\}$, where each school has one seat. Tables below describe schools' ordinal priority lists (which are induced by their score vectors), students' preferences, and the corresponding student-proposing DA procedure.

[5] Kesten (2010) shows that under a limited information setting, truth-telling is an ordinal Bayes Nash equilibrium in the preference revelation game under EADAM.

s_1	s_2	s_3	1	2	3	4
1	2	3	s_2	s_1	s_1	s_2
2	4	1	s_3	s_2	s_3	$\emptyset$
3	1	$\vdots$	s_1	$\emptyset$	$\vdots$	
$\vdots$	$\vdots$					

s_1	s_2	s_3	$\emptyset$
$\boxed{2}, 3$	$1, \boxed{4}$		
		$1, \boxed{3}$	
$\boxed{1}, 2$			
	$\boxed{2}, 4$		
$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$

Suppose the difference between the scores of students 2 and 3 at school s_1 is $a_2^{s_1} - a_3^{s_1} = 1$. Similarly, suppose $a_4^{s_2} - a_1^{s_2} = 3$. At the round-0 DA outcome, student 1 can either exchange her seat at s_1 with student 2 to obtain a seat at s_2, or to exchange with student 3 to obtain a seat at s_3, creating different intensities of priority violation.

Case I: $\varepsilon = 2$. In round-1 of the ε-EADAM, student 4 will be removed with her assigned underdemanded school $\emptyset$. Since 4 still desires s_2 and $a_4^{s_2} - a_1^{s_2} > \varepsilon$, student 1 will not be allowed to apply for s_2 any more. In round-1 DA, 2 matches to the underdemanded school s_2, hence we can remove them and proceed to round-2. The round-1 and round-2 DA procedures are as follows:

Round-1 DA:

s_1	s_2	s_3
$\boxed{2}, 3$		$\boxed{1}$
		$1, \boxed{3}$
$\boxed{1}, 2$		
$\boxed{1}$	$\boxed{2}$	$\boxed{3}$

Round-2 DA:

s_1	s_3
$\boxed{3}$	$\boxed{1}$

Case II: $\varepsilon = 4$. In round-1 of the ε-EADAM, we don't need to remove student 1's application to s_2 after student 4 is removed with $\emptyset$, as now $a_4^{s_2} - a_1^{s_2} < \varepsilon$. Therefore, round-1 DA becomes

Round-1 DA:

s_1	s_2	s_3
$\boxed{2}, 3$	$\boxed{1}$	
		$\boxed{3}$
$\boxed{2}$	$\boxed{1}$	$\boxed{3}$

In sum, the ε-EADAM outcomes are as follows:

$\varepsilon = 2 : \begin{array}{c|c|c|c} s_1 & s_2 & s_3 & \emptyset \\ \hline \boxed{3} & \boxed{2} & \boxed{1} & \boxed{4} \end{array}$ $\quad$ $\varepsilon = 4 : \begin{array}{c|c|c|c} s_1 & s_2 & s_3 & \emptyset \\ \hline \boxed{2} & \boxed{1} & \boxed{3} & \boxed{4} \end{array}$

Hence as ε increases, student 3 becomes worse off. At the DA matching, when $\varepsilon = 2$, ε-EADAM selects the trading cycle between students 1 and 3, while when ε increases to 4, it selects the trading cycle between students 1 and 2 (which is not eligible when $\varepsilon = 2$). The key insight is that increase in ε makes more trading eligible, hence potentially creates more competition among students in order to get improved.

Remark 1. *Although ε-EADAM does not make all students weakly better off as ε increases, there always exists a selection of ε-constrained efficient matchings which does that. Suppose the values of ε are integers. For any preference profile of students, such a selection can be defined inductively. If for some ε, a matching μ_ε is the selected ε-constrained efficient matching and $\varepsilon' = \varepsilon + 1$. If the ε'-EADAM outcome weakly Pareto dominates μ_ε, then select it as $\mu_{\varepsilon'}$. Otherwise, as μ_ε is also ε'-stable, there must exist an ε'-constrained efficient matching that weakly Pareto dominates $\mu_{\varepsilon'}$, select it as $\mu_{\varepsilon'}$.*

4 Conclusion

The ε-stability and the ε-EADAM we propose offer an alternative method on how to relax the stability constraint and improve students' welfare. The concept and mechanism are useful in real life practices where schools assign scores to determine students' priorities. Such cardinal information in priorities is often public, which makes the intensity of priority violation measurable and observable. Therefore, it allows us to choose as a policy variable how much priority violation can be permitted. The choice of ε may vary across schools and across applications, based the trade-off between the loss of stability and the potential gain in students' welfare. Whether to relax stability and by how much is then left for debate among practitioners.

5 Appendix

A. Proof of Theorem 1

For each $k \geq 0$, let P^k denote the updated preferences of students who still remain in round-k DA of the ε-EADAM procedure, and let matching α_k denote the matching produced by round-k of the algorithm, in which the students who remain after removal in round-k are matched by round-k DA, and students removed in or before round-k are matched with the seats they are removed together with. Let α denote the eventual matching produced by the algorithm.

We begin by showing that the eventual matching α produced by the ε-EADAM is ε-stable. Consider any student i, and suppose i is matched with an underdemanded school in round-k DA of the ε-EADAM procedure. Suppose student j is removed earlier than i or at the same step as i, then at the step that j is removed, either j's assignment is still in i's preference or it has been removed from i's preference at some earlier step. In the former case, apparently i does not desire the assignment of j since j's assignment is underdemanded, and thus j does not violate i's priority. In the latter case, note that the assignment of j is in j's preference but not in i's preference. It implies that j has higher scores at this school than i, and thus j does not violate i's priority.

If student j is removed later than i and i desires j's assignment s, then at the step that i is removed, in the algorithm, s would have been removed from P_j if $a_j^s < a_i^s - \varepsilon$. Therefore, j cannot violate i's priority by more than ε. As a result, the matching produced by ε-EADAM is ε-stable.

Next, we show that any matching μ that Pareto dominates α must not be ε-stable. Since α weakly Pareto dominates all α_k's, we know that μ Pareto dominates all α_k's. For each $k \geq 0$, let $UD_k \subset I$ denote the set of students matched with underdemanded schools at the round-k DA matching $DA(P^k, \succ)$. Since μ Pareto dominates the round-0 outcome α_0, for all $i \in UD_0$, $\mu(i) = \alpha_0(i)$.

We use the following lemma to carry out induction.

Lemma 2. *Let μ be an ε-stable matching. Consider any round-k, $k \geq 1$, of the ε-EADAM. Suppose we already know that for each $0 \leq l \leq k-1$, μ Pareto dominates α_l implies that for all $i \in UD_l, \mu(i) = \alpha_l(i)$. Then μ Pareto dominates α_k implies that for all $i \in UD_k, \mu(i) = \alpha_k(i)$.*

Proof. Suppose for some $i \in UD_k$, $\mu(i) P_i \alpha_k(i)$. Since μ Pareto dominates α_k under P, μ also Pareto dominates α_l for each $0 \leq l \leq k-1$ due to Lemma 1. By the assumption of the lemma, $\mu(i) = \alpha_l(i)$, for each $i \in UD_l$ and each $0 \leq l \leq k-1$. Therefore, μ Pareto dominates α_k implies that for students who remain after removal in round-k, μ Pareto dominates $DA(P^k, \succ)$ under P. So either μ also Pareto dominates $DA(P^k, \succ)$ under P^k or for some i' who remains in round-k, $\mu(i')$ is removed from $P^k_{i'}$ before round-k. In the former case, we know that for all $i \in UD_k$, $\mu(i) = DA(P^k, \succ)(i) = \alpha_k(i)$. This contradicts with the assumption that $\mu(i) P_i \alpha_k(i)$ for some $i \in UD_k$. In the latter case, μ matches some student who remains after removal in round-k with a school removed in the earlier modifications of her preference. Again by definition of the algorithm, μ is not ε-stable. We have a contradiction.

□

Since the case of $k = 1$ holds, by induction (due to lemma 2), for each $k \geq 1$, if μ Pareto dominates α_k and is ε-stable (under the original preference profile P), then $i \in UD_k$ implies $\mu(i) = \alpha_k(i)$.

We now prove the theorem. Suppose there exists a matching μ that Pareto dominates α and is ε-stable. Let round-K be the last round of the ε-EADAM, which produces the eventual outcome $\alpha = \alpha_K$. Since μ Pareto dominates α, it Pareto dominates $\alpha_0, \ldots, \alpha_K$. Since μ is ε-stable, due to Lemma 3, for each $0 \leq k \leq K$, if $i \in UD_k$, then $\mu(i) = \alpha_k(i)$. That is, at matching μ, each student $i \in I$ is matched with the seat she is removed together with. Therefore, $\mu = \alpha$. We have a contradiction.

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