Independence systems in gross-substitute valuations

Chao Huang[*]

September 30, 2018

Abstract

Objects may exhibit substitutabilities, complementarities as well as independencies for agents. In this paper we show that under the Kelso-Crawford gross substitutes condition, the sets of mutual independent objects form a *matroid*. Hence the structure of independent objects under the gross substitutes condition has much in common with the *independence system* of a graph or a vector space.

Keywords: Gross substitutes; matching; combinatorial auctions; matroids; $M^\natural$-concave function

JEL classification: C78, D46

1 Introduction

The gross substitutes (henceforth, GS) condition introduced by Kelso and Crawford (1982) with its variants has played a critical role for the existence of core, equilibrium and stable

[*]Institute for Social and Economic Research, Nanjing Audit University. Email: tommyhuang16@163.com. I am grateful to an anonymous referee, Kazuo Murota, Ning Sun and Qianfeng Tang for very helpful comments and suggestions. All errors are mine.

matching in various economic settings with indivisible goods. A valuation satisfies the GS condition if raising the prices of some objects does not decrease the demand of other objects. The GS condition has also been characterized in many works (see e.g. Gul and Stacchetti, 1999, Reijinierse et al., 2002, Fujishige and Yang, 2003).

Objects may exhibit substitutabilities, complementarities as well as independencies for agents. Independencies among objects generally exist in real life. For example, goods from different categories are probably independent for the agent: an apple hardly exhibit any substitutability or complementarity to a pair of shoes; the demand for a table lamp is probably independent with that for a bicycle. So far attentions have been paid to substitutabilities and complementarities among objects, little is known about the independent objects. In this paper we show that under the GS condition, the sets of mutual independent objects form a special structure. Consider the following example, suppose there are three objects x, y, and z. v is a valuation such that

$$\begin{gathered} v(\{x\}) = 2, \quad v(\{y\}) = 2, \quad v(\{z\}) = 3 \\ v(\{x,y\}) = 4, \quad v(\{y,z\}) = t, \quad v(\{x,z\}) = 4 \\ v(\{x,y,z\}) = 5 \end{gathered}$$

Since $v(\{x,y\}) = v(\{x\}) + v(\{y\})$, objects x and y exhibit neither substitutability nor complementarity. Thus objects in the set $\{x,y\}$ are mutually independent for the agent. Moreover, mutual independence among objects in a set X should require that for every $Y \subseteq X$, $v(Y)$ is equal to the sum of $v(\{y\})$ over all elements y of Y. In this example, $\{x,y\}$ is also a maximal set in which the objects are mutually independent. This is easy to see since in the unique superset $\{x,y,z\}$, x and z are obviously dependent, and also $v(\{x,y,z\}) \neq v(\{x\}) + v(\{y\}) + v(\{z\})$. The main result in this paper implies that, under the GS condition, the maximal sets in which the objects are mutual independent have the same cardinality. Such maximal sets in this example are $\{x,y\}$, $\{z\}$ when $t \neq 5$, and $\{x,y\}$, $\{y,z\}$ when $t = 5$. By our result, we know that v is not a GS valuation if $t \neq 5$. The readers can easily check that v satisfies the GS condition if and only if $t = 5$[1].

In particular, we show that for a GS valuation, the sets of mutual independent objects form a *matroid*. A matroid is a structure that abstracts and generalizes the notion of independence from linear algebra. Surprisingly our result indicates that the structure of independent objects under a GS valuation has much in common with the *independence system* (see Definition 2) of a graph or a vector space. The proof of our main theorem takes advantage of the characterizing result provided by Fujishige and Yang (2003). Recent studies show that the tools from matroid theory and discrete convex analysis are very useful in various fields such as dynamic auction (Gul and Stacchetti, 2000, Murota et al., 2016), two-sided matching (Fujishige and Tamura, 2007, Ostrovsky and Paes Leme, 2015, Kojima et al, 2017) and resource allocation (Roth et al., 2005). Excellent surveys have been made by Murota (2016) and Paes Leme (2017).

Since there are different characterizations for the GS condition, and also terminologies in different subjects equivalent to GS valuation, our result can be proved in different ways. For example, it is also convenient to prove the result with the no complementarities condition of Gul and Stacchetti (1999), or with some existing results in discrete convex analysis. However, to our best knowledge we are the first to find this matroid structure in economic valuations[2]. We believe our result is interesting for its connection between the independency in economic sense and the independence system in combinatorial mathematics.

2 Model

There is a finite set S of objects with cardinality n. An agent has a valuation function $v : 2^S \to \mathbb{R}$, that specifies the value she gets from each possible subset of objects. It

[1] A convenient method provided by Reijinierse et al. (2002) can be used to check GS condition.

[2] The author have consulted Kazuo Murota. He says that he has never seen this statement in the literature of discrete convex analysis. We agree that our result can also be derived from some existing results. We adopt the current proof since it requires only elementary mathematical argument, which is more accessible to readers.

is assumed that $v(\emptyset) = 0$ and v is a nondecreasing function, i.e., $X \subseteq Y \subseteq S$ implies $v(X) \leq v(Y)$. These two assumptions also imply that $v(X) \geq 0$ for every $X \subseteq S$. A valuation v is *sub-additive* if it satisfies: $v(X) + v(Y) \geq v(X \cup Y)$ for all sets $X, Y \subseteq S$.

Given a price vector $p \in \mathbb{R}^n$ and a bundle of objects $X \subseteq S$, denote by $p(X) = \sum_{i \in X} p_i$ the price of bundle X. The agent has to solve the following decision problem:

$$\max_{X \subseteq S} \{v(X) - p(X)\}$$

The *demand correspondence* $D(p)$ is the set of solutions to the problem. A valuation function v satisfies the *gross substitutes* (GS) condition if increasing the price of an object will not decrease the demand for any other object.

Definition 1. *A valuation function v satisfies the gross substitutes condition if for any two price vector p and p' such that $p \leq p'$ and any $X \in D(p)$, there exists bundle $X' \in D(p')$ such that such that $\{i \in X \mid p_i = p'_i\} \subseteq X'$.*

Every GS valuation is sub-additive. If $X \subset S$, and $y \in S \setminus X$, we shortly write $X \cup y \equiv X \cup \{y\}$; and if $x \in X$, we shortly write $X \setminus x \equiv X \setminus \{x\}$. Furthermore, if v is a valuation and $x_1, x_2, \ldots, x_k$ are objects, we shortly write $v(x_1, x_2, \ldots, x_k) \equiv v(\{x_1, x_2, \ldots, x_k\})$.

3 The independence system

A *set system* is a pair $(S, \mathcal{F})$ where S is a finite set of objects (called the *ground set* of the set system) and $\mathcal{F} \subseteq 2^S$ a collection of subsets of set S. An *independence system* is a special set system defined as follows.

Definition 2. *An independence system is a pair $(S, \mathcal{I})$, where S is a finite set of objects and $\mathcal{I}$ is a collection of subsets of set S (called the independent sets of the independence system) such that the following statements hold:*

(I1) $\emptyset \in \mathcal{I}$

(I2) *If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.*

Statement **(I1)** says that the empty set is independent. Statement **(I2)** says that any subset of an independent set is independent. Non-elements of $\mathcal{I}$ are called *dependent sets*.

Let v be a valuation and $\mathcal{I} = \{X|v(X) = \sum_{x\in X} v(x), X \subseteq S\}$. $\mathcal{I}$ is a collection of subsets of S such that the valuation of the set is equal to the sum of the valuations of each object in the set. It is shown that $(S, \mathcal{I})$ forms an independence system when v is sub-additive.

Lemma 1. *Suppose v is a sub-additive valuation and $\mathcal{I} = \{X|v(X) = \sum_{x\in X} v(x), X \subseteq S\}$, then $(S, \mathcal{I})$ is an independence system.*

Proof. Statement (I1) of Definition 2 is clearly satisfied. For any $X \in \mathcal{I}$ and $Y \subseteq X$, since v is a sub-additive valuation, we have , $v(Y) \leq \sum_{y\in Y} v(y)$. Suppose $v(Y) < \sum_{y\in Y} v(y)$. Since $v(X \setminus Y) \leq \sum_{x\in X\setminus Y} v(x)$ and $v(X) \leq v(X \setminus Y) + v(Y)$, we have $v(X) < \sum_{x\in X} v(x)$ and thus $X \notin \mathcal{I}$. A contradiction. Hence, $v(Y) = \sum_{y\in Y} v(y)$ and $Y \in \mathcal{I}$. $\square$

Remark 1. Lemma 1 does not rely on the assumption that $v(\emptyset) = 0$ or v is nondecreasing. However, it does not hold when the range of valuation includes $-\infty$, which is allowed in the literature of $M^\natural$-concave functions (Murota and Shioura, 1999). For instance, suppose $v : 2^S \to \mathbb{R} \cup \{-\infty\}$ is a set function such that $v(x) = 1$, $v(y) = 1, v(z) = -\infty, v(x, y) = 1, v(y, z) = y(x, z) = v(x, y, z) = -\infty$. Then $v(x, y, z) = v(x) + v(y) + v(z)$ and $v(x, y) < v(x) + v(y)$, while v is sub-additive.

4 The underlying matroid

A *matroid* is a special independence system. An independence system forms a matroid if it satisfies an extra independence augmentation axiom.

Definition 3. *The independence system $(S, \mathcal{I})$ is called a matroid if it satisfies the following condition:*

(I3) *If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there exists object $y \in Y \setminus X$ such that $X \cup y \in \mathcal{I}$.*

In this paper, we will show that for a GS valuation, the independent sets of the objects form a matroid. We will make use of tools from discrete convex analysis. Discrete convex analysis is a theory developed by Murota (1998, 2003) that combines the ideas in continuous optimization and combinatorial optimization for nonlinear discrete optimization. We now introduce the $M^\natural$-concave functions[3], due to Murota and Shioura (1999).

Definition 4. *A valuation function v is an $M^\natural$-concave function if for each $X, Y \subseteq S$ and $x \in X \setminus Y$,*

$$v(X) + v(Y) \leq \max[v(X \setminus x) + v(Y \cup x), \max_{y \in Y \setminus X} \{v(X \cup y \setminus x) + v(Y \cup x \setminus y)\}]$$

Fujishige and Yang (2003) made the following observation.

Lemma 2. *(Fujishige and Yang, 2003) A valuation function v satisfies the gross substitutes condition if and only if it is $M^\natural$-concave.*

Now we are ready to prove our main result.

Theorem 1. *Suppose v is a gross-substitute valuation and $\mathcal{I} = \{X | v(X) = \sum_{x \in X} v(x), X \subseteq S\}$, then $(S, \mathcal{I})$ is a matroid.*

Proof. Let $\mathcal{B}$ be the collection of maximal elements in $\mathcal{I}$ with respect to set inclusion. For each $B_1, B_2 \subseteq \mathcal{B}$, we have $B_1 \setminus B_2 \neq \emptyset$. By Lemma 2, for each $x \in B_1 \setminus B_2$,

$$v(B_1) + v(B_2) \leq \max[v(B_1 \setminus x) + v(B_2 \cup x), \max_{y \in B_2 \setminus B_1} \{v(B_1 \cup y \setminus x) + v(B_2 \cup x \setminus y)\}] \quad (4.1)$$

By Lemma 1, $(S, \mathcal{I})$ is an independence system, thus $B_1 \setminus x \in \mathcal{I}$ and we have $v(B_1 \setminus x) = \sum_{z \in B_1 \setminus x} v(z)$. Since v is a GS valuation and $B_2 \cup x \notin \mathcal{I}$ as B_2 is a maximal element in $\mathcal{I}$, we

[3]The symbol $\natural$ is read "natural".

have $v(B_2 \cup x) < \sum_{z \in B_2 \cup x} v(z)$, then

$$\begin{aligned} v(B_1) + v(B_2) \quad &= \textstyle\sum_{z \in B_1} v(z) + \sum_{z \in B_2} v(z) \\ &= \textstyle\sum_{z \in B_1 \setminus x} v(z) + \sum_{z \in B_2 \cup x} v(z) \\ &> v(B_1 \setminus x) + v(B_2 \cup x) \end{aligned}$$

Thus the maximum of 4.1 corresponds to the second expression. Since for any $y \in B_2 \setminus B_1$,

$$v(B_1) + v(B_2) = \sum_{z \in B_1} v(z) + \sum_{z \in B_2} v(z) \geq v(B_1 \cup y \setminus x) + v(B_2 \cup x \setminus y)$$

4.1 holds only if for each B_1, $B_2 \subseteq \mathcal{B}$, and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 \cup y \setminus x \in \mathcal{I}$ and $B_2 \cup x \setminus y \in \mathcal{I}$.

We then show it must be that $B_1 \cup y \setminus x \in \mathcal{B}$. Suppose $B_1 \cup y \setminus x \notin \mathcal{B}$, there exists $B_3 = \{B_1 \cup y \setminus x\} \cup Z \in \mathcal{B}$, such that$\emptyset \neq Z \subseteq S$ and $Z \cap \{B_1 \cup y \setminus x\} = \emptyset$. Also $x \notin Z$, otherwise B_1 is not a maximal element in $\mathcal{I}$ since $B_1 \subset B_3 \in \mathcal{I}$. Then the argument in above paragraph implies that for B_1, B_3 and x, since $x \in B_1 \setminus B_3$, there exists $u \in B_3 \setminus B_1$ such that $B_3 \cup x \setminus u \in \mathcal{I}$. Because $B_3 = \{B_1 \cup y \setminus x\} \cup Z$, we have $B_3 \cup x \setminus u = B_1 \cup y \cup Z \setminus u \in \mathcal{I}$. Since $u \in B_3 \setminus B_1 = y \cup Z$, $Z \neq \emptyset$ and $y \notin Z$ (this is because $Z \cap \{B_1 \cup y \setminus x\} = \emptyset$), we know that $y \cup Z$ contains at least two elements and $B_1 \subset B_1 \cup y \cup Z \setminus u \in \mathcal{I}$. Thus B_1 is not a maximal element in $\mathcal{I}$, a contradiction.

Therefore, we have shown that for each B_1, $B_2 \subseteq \mathcal{B}$, and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 \cup y \setminus x \in \mathcal{B}$. The theorem then follows immediately (see e.g. Theorem 1.2.3 of Oxley, 1992).

□

Remark 2. It is natural to imagine that under the GS condition, the independent objects come from different categories of commodities. That is to say, there is a partition of the objects $\{S_1, S_2, \ldots, S_k\}$ that represents the categories of the goods. Objects in the same S_i are substitutes in the sense that the valuation within each S_i is strictly sub-additive. Any set X such that $X \cap S_i$ contains at most one object for each i is an independent set. For instance, in our illustrative example with $t = 5$, there is a partition $\{x, z\}, \{y\}$ that divides the objects into two categories. The independent sets then appear in the above manner. However, the independence system of a GS valuation can be different from the above structure as it can be some other type of matroid[4]. To see this, consider a zoo that owns two vacant cages and has the following valuation on a lion(l), a wolf(w) and a bear(b): $v(l) = v(w) = v(b) = 1$, $v(l, w) = v(w, b) = v(l, b) = v(l, w, b) = 2$. While this valuation satisfies the GS condition, a partition as above on the animals is impossible.

References

Fujishige, S., Tamura, A., 2007. A two-sided discrete-concave market with possibly bounded side payments: An approach by discrete convex analysis. *Math. Oper. Res.*, 32, 136-155.

Fujishige, S., Yang, Z., 2003. A note on Kelso and Crawford's gross substitutes condition. *Math. Oper. Res.*, 28, 463-469.

Gul, F., Stacchetti, E., 1999. Walrasian equilibrium with gross substitutes. *J.Econ. Theory*, 87, 95-124.

Gul, F., Stacchetti, E., 2000. The English auction with differentiated commodities. *J. Econ. Theory*, 92, 66-95.

Kelso, A. S., Crawford, V.P., 1982. Job matching, coalition formation and gross substitutes. *Econometrica*, 50, 1483-1504.

Kojima, F., Tamura, A., Yokoo, M., 2017. Designing matching mechanisms under constraints: an approach from discrete convex analysis. Mimeo.

Murota, K., 1998. Discrete convex analysis. *Math. Programming*, 83, 313-371.

[4] I thank the referee for this insightful observation.

Murota, K., 2003. Discrete convex analysis. SIAM, Philadelphia.

Murota, K., 2016. Discrete convex analysis: a tool for economics and game theory. *J. of Mech. and Inst. Design*, 1(1), 151-273.

Murota, K., Shioura, A., 1999. M-convex function on generalized polymatroid. *Math. Oper. Res.*, 24, 95-105.

Murota, K., Shioura, A., Yang, Z., 2016. Time bounds for iterative auctions: A unified approach by discrete convex analysis. *Discrete Optim.*, 19, 36-62.

Ostrovsky, M., Paes Leme, R., 2015. Gross substitutes and endowed assignment valuations. *Theor. Econ.*, 10, 853-865.

Oxley, J.G., 1992. Matroid Theory. Oxford University Press, New York.

Paes Leme, R., 2017. Gross substitutability: An algorithm survey. *Games Econ. Behav.*, 106, 294-316.

Reijinierse, H., Gellekom, A., Potters, J.A.M., 2002. Verifying gross substitutability. *Econ. Theory*, 20, 767-776.

Roth, A.E., Sönmez, T., Ünver, M.U., 2005. Pairwise kidney exchange. *J. Econ. Theory*, 125(2), 151-188.